7.3 ⋆ $\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}k(x^2 + y^2)$ and the two Lagrange equations are $-kx = m\ddot{x}$ and $-ky = m\ddot{y}$. In the general solution, x and y oscillate with the same angular frequency $\omega = \sqrt{k/m}$ and the point (x, y) moves around an ellipse.

7.8 ⋆⋆ **(a)** $\mathcal{L} = T - U = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}kx^2$.

(b) Solving for x_1 and x_2 in terms of X and x, we find

$$x_1 = X + \tfrac{1}{2}x + \tfrac{1}{2}l \qquad \text{and} \qquad x_2 = X - \tfrac{1}{2}x - \tfrac{1}{2}l.$$

Differentiating these, and substituting into $\mathcal{L}$, we find

$$\mathcal{L} = \tfrac{1}{2}m[(\dot{X} + \tfrac{1}{2}\dot{x})^2 + (\dot{X} - \tfrac{1}{2}\dot{x})^2] - \tfrac{1}{2}kx^2 = m\dot{X}^2 + \tfrac{1}{4}m\dot{x}^2 - \tfrac{1}{2}kx^2.$$

The two Lagrange equations are

$$\text{X eqn:} \quad \frac{\partial \mathcal{L}}{\partial X} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}} \qquad \text{or} \qquad 0 = 2m\ddot{X}$$

and

$$\text{x eqn:} \quad \frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \qquad \text{or} \qquad -kx = \tfrac{1}{2}m\ddot{x}.$$

(c) The X equation implies that $\dot{X}(t) = \text{const} = V_\text{o}$ and hence that $X(t) = V_\text{o}t + X_\text{o}$; that is, the CM moves like a free particle, which we could have anticipated, since there are no external forces. The x equation has the general solution $x(t) = A\cos(\omega t - \delta)$; that is, the two masses oscillate in and out, relative to each other, with frequency $\omega = \sqrt{2k/m}$. The factor of $2k$ inside the square root, can be understood in several ways; for example, the spring is compressed (or stretched) by twice the amount that either separate mass moves. Thus the force on either mass is as if the spring had force constant $2k$.

7.10 ⋆ $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = \rho/\tan\alpha$, and, in the other direction, $\rho = \sqrt{x^2 + y^2}$ (or $\rho = z\tan\alpha$) and $\phi = \arctan(y/x)$ with ϕ chosen to lie in the right quadrant.

7.12 ⋆ If we define $\mathcal{L} = T - U = \frac{1}{2}m\dot{x}^2 - U(x)$, then $\partial\mathcal{L}/\partial x = -\partial U/\partial x = F$ and $(d/dt)(\partial\mathcal{L}/\partial\dot{x}) = m\ddot{x}$. Substituting into Newton's second law $F + F_\text{fric} = m\ddot{x}$, we find that

$$\frac{\partial \mathcal{L}}{\partial x} + F_\text{fric} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}\,.$$

7.14 ⋆ Recalling that $I = \frac{1}{2}mR^2$ and that $\omega = \dot{x}/R$, we see that the kinetic energy is $T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\omega^2 = \frac{3}{4}m\dot{x}^2$. Therefore, the Lagrangian is $\mathcal{L} = \frac{3}{4}m\dot{x}^2 + mgx$, the Lagrange equation is $mg = 3m\ddot{x}/2$, and $\ddot{x} = 2g/3$.

7.16 ⋆ Since $\omega = v/R$, the cylinder's KE is $T = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}(m + I/R^2)\dot{x}^2$. The PE is $U = -mgx\sin\alpha$, so the Lagrangian is $\mathcal{L} = \frac{1}{2}(m + I/R^2)\dot{x}^2 + mgx\sin\alpha$ and the Lagrange equation is $mg\sin\alpha = (m + I/R^2)\ddot{x}$. Therefore $\ddot{x} = (mg\sin\alpha)/(m + I/R^2)$.

7.20 ⋆ The helix satisfies $\rho = R$ and $z = \lambda\phi$. Thus the bead's velocity is $\mathbf{v} = (\dot{\rho}, \rho\dot{\phi}, \dot{z}) = (0, R\dot{\phi}, \dot{z}) = \dot{z}(0, R/\lambda, 1)$ and its KE is $\frac{1}{2}mv^2 = \frac{1}{2}m\dot{z}^2(1 + R^2/\lambda^2)$. The PE is $U = mgz$, so the Lagrangian is $\mathcal{L} = T - U = \frac{1}{2}m\dot{z}^2(1 + R^2/\lambda^2) - mgz$. The Lagrange equation is $-g = (1 + R^2/\lambda^2)\ddot{z}$ (after canceling a factor of m), and $\ddot{z} = -g/(1 + R^2/\lambda^2)$. When $R \to 0$ this answer reduces to $\ddot{z} = -g$, which is correct because in this limit the helix reduces to a vertical frictionless wire, on which the acceleration is just g vertically down.

7.22 ⋆ We must first write down the Lagrangian in an inertial frame, for which the natural choice is a frame fixed to the earth, relative to which the elevator is accelerating upward. The point of support in the elevator's ceiling has velocity $\mathbf{V} = (0, at)$ (if we measure x horizontally and y vertically up) and position $(0, \frac{1}{2}at^2)$. The bob's velocity relative to the elevator is $\mathbf{v}_{\text{rel}} = (l\dot{\phi}\cos\phi, l\dot{\phi}\sin\phi)$. Thus its velocity relative to the ground is $\mathbf{v} = \mathbf{V} + \mathbf{v}_{\text{rel}} = (l\dot{\phi}\cos\phi, at + l\dot{\phi}\sin\phi)$. The bob's height above the ground is $y = \frac{1}{2}at^2 - l\cos\phi$. You can now write down the KE and PE and (after a little algebra) the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}mv^2 - mgy = \tfrac{1}{2}m\left(a^2t^2 + 2atl\dot{\phi}\sin\phi + l^2\dot{\phi}^2\right) - mg\left(\tfrac{1}{2}at^2 - l\cos\phi\right).$$

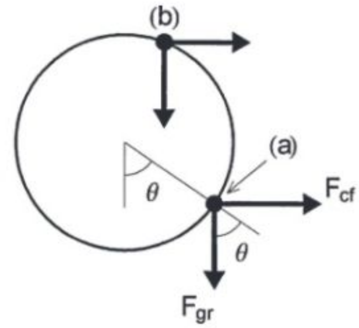
The Lagrange equation is

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\phi} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} \quad\Longrightarrow\quad matl\dot{\phi}\cos\phi - mgl\sin\phi &= \frac{d}{dt}(ml^2\dot{\phi} + matl\sin\phi)\\ &= ml^2\ddot{\phi} + matl\dot{\phi}\cos\phi + mal\sin\phi.\end{aligned}$$

Making a couple of cancelations and rearranging, we arrive at the equation $l\ddot{\phi} = -(g+a)\sin\phi$, which is the equation for a normal (non-accelerating) pendulum, except that g has been replaced by $(g + a)$.

7.26 ⋆ From Eq.(7.79), $\Omega'^2 = \omega^2\sin^2\theta_o = \omega^2(1-\cos^2\theta_o)$, and, from (7.76), $\cos\theta_o = g/(\omega^2 R)$. Combining these, we find that $\Omega' = \sqrt{\omega^2 - g^2/(\omega R)^2}$ as claimed.

7.28 ⋆⋆ **(a)** Consider the equilibrium point with $0 < \theta < 90°$, labeled (a) in the picture. As seen in the rotating frame, the bead is subject to three forces, the normal force of the hoop (not shown in the picture), the force of gravity, $\mathbf{F}_{\rm gr} = m\mathbf{g}$, and the centrifugal force $\mathbf{F}_{\rm cf} = m\omega^2\rho$, radially out from the axis of rotation, where $\rho = R\sin\theta$ is the distance of the bead from the axis. The bead will be in equilibrium if and only if the tangential component of the net force is zero. Since the tangential component of the normal force is zero, this condition is,



$$F_{\rm tang} = -(mg)\sin\theta + (m\omega^2 R\sin\theta)\cos\theta = m(\omega^2 R\cos\theta - g)\sin\theta = 0.$$

This condition is satisfied if and only if $\cos\theta = g/\omega^2 R$, which is precisely the condition (7.71).

(b) Suppose the bead has moved a little away from the equilibrium at the top of the hoop, as indicated by (b) in the picture. At this position the tangential components of $\mathbf{F}_{\rm gr}$ and $\mathbf{F}_{\rm cf}$ are both pulling the bead away from equilibrium. Therefore the equilibrium at the top is definitely unstable.

(c) Consider the equilibrium with θ negative [across from (a) in the picture] and suppose the bead moves a little up from the equilibrium (θ more negative). This makes $\cos\theta$ smaller, and the first parenthesis on the right of Eq.(7.73) becomes negative. Since $\sin\theta$ is also negative, $\ddot\theta$ is positive, and the bead accelerates back toward equilibrium. Similarly, if the bead moves down from equilibrium, $\ddot\theta$ becomes negative and, again, the bead accelerates back toward equilibrium. Therefore, the equilibrium is stable.

7.29 ⋆⋆ Because the angle between the line OP and the horizontal is ωt, the position of P is $(R\cos\omega t, R\sin\omega t)$. Therefore the position of the pendulum's bob is

$$\mathbf{r} = (x, y) = (R\cos\omega t + l\sin\phi, R\sin\omega t - l\cos\phi)$$

and its velocity is

$$\mathbf{v} = (\dot x, \dot y) = (-\omega R\sin\omega t + \dot\phi l\cos\phi, \omega R\cos\omega t + \dot\phi l\sin\phi).$$

Therefore the Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}mv^2 - mgy = \tfrac{1}{2}m[\omega^2R^2 + \dot\phi^2 l^2 + 2\omega R\dot\phi l\sin(\phi - \omega t)] - mg(R\sin\omega t - l\cos\phi)$$

where I have used a couple of trig identities to combine various terms (and omitted a constant). The two derivatives of $\mathcal{L}$ are

$$\frac{\partial\mathcal{L}}{\partial\phi} = m\omega R\dot\phi l\cos(\phi - \omega t) - mgl\sin\phi \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial\dot\phi} = m[\dot\phi l^2 + \omega Rl\sin(\phi - \omega t)]$$

and the Lagrange equation, after a couple of cancellations, is

$$l\ddot\phi = -g\sin\phi + \omega^2 R\cos(\phi - \omega t).$$

As $\omega \to 0$, this becomes $l\ddot\phi = -g\sin\phi$, the equation for an ordinary simple pendulum.

7.34 ⋆⋆ **(a)** Let the unstretched length of the spring be l and consider a short segment of spring a distance ξ from the fixed end and of length $d\xi$. Since the spring is uniform, the mass of this segment is $Md\xi/l$ and since the spring stretches uniformly its velocity (when the cart has velocity $\dot{x}$) is $\dot{x}\xi/l$. Therefore the KE of this segment is $\frac{1}{2}M\dot{x}^2\xi^2 d\xi/l^3$, and the total KE of the whole spring is

$$T_{\text{spr}} = \frac{1}{2}\frac{M\dot{x}^2}{l^3}\int_0^l \xi^2 d\xi = \frac{1}{6}M\dot{x}^2$$

Therefore the Lagrangian for the system of spring and cart is $\mathcal{L} = \frac{1}{2}(m + M/3)\dot{x}^2 - \frac{1}{2}kx^2$.

(b) The Lagrange equation is $-kx = (m + M/3)\ddot{x}$, which is the same as for the usual massless spring except that m, the mass of the cart, has been replaced by $m + M/3$. In particular, the angular frequency of the oscillations is $\omega = \sqrt{k/(m + M/3)}$.

7.37 ⋆⋆⋆ **(a)** The hanging mass is a distance $L - r$ below the table. Thus it's KE is $\frac{1}{2}m\dot{r}^2$ and its PE is $-mg(L - r)$, or just mgr, if we drop an uninteresting constant. The mass on the table has KE $= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$ and PE which is constant and we may as well take to be zero. Thus

$$\mathcal{L} = m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\phi}^2 - mgr.$$

(b) The two Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \qquad \text{or} \qquad mr\dot{\phi}^2 - mg = 2m\ddot{r}$$

and

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \qquad \text{or} \qquad 0 = \frac{d}{dt}(mr^2\dot{\phi}).$$

The ϕ equation says simply that the angular momentum $\ell = mr^2\dot{\phi}$ is constant.

(c) Clearly $\dot{\phi} = \ell/mr^2$, and the r equation can be rewritten as

$$2m\ddot{r} = \frac{\ell^2}{mr^3} - mg. \qquad \text{(vii)}$$

The length r can remain constant if and only if $\ddot{r} = 0$. This requires that $\ell^2/mr_o{}^3 = mg$. (This condition says that the centripetal force needed to keep the upper mass in a circular path must equal the tension needed to hold the lower mass at a fixed height.) Therefore, $r_o = [\ell^2/(m^2 g)]^{1/3}$

(d) If $r = r_o + \epsilon$, then (vii) becomes

$$2m\ddot{\epsilon} = \frac{\ell^2}{m(r_o + \epsilon)^3} - mg = \frac{\ell^2}{mr_o^3}\left(1 + \frac{\epsilon}{r_o}\right)^{-3} - mg \approx \frac{\ell^2}{mr_o^3}\left(1 - 3\frac{\epsilon}{r_o}\right) - mg$$

where, to get the final expression, I used the binomial approximation. The first and last terms in the final expression cancel, and we are left with $2m\ddot{\epsilon} = -3(\ell^2/mr_o{}^4)\epsilon$, which implies that ϵ oscillates sinusoidally with frequency $\sqrt{3/2}\,\ell/mr_o^2 = \sqrt{3g/2r_o}$, since $\ell/m = \sqrt{gr_o^3}$. In particular, since the displacement ϵ oscillates, the equilibrium at $r = r_o$ is stable.

7.40 ⋆⋆⋆ **(a)** The bob's velocity is $\mathbf{v} = (0, R\dot{\theta}, R\dot{\phi}\sin\theta)$ and its height below the support is $z = R\cos\theta$. Therefore

$$\mathcal{L} = T - U = \tfrac{1}{2}mR^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + mgR\cos\theta$$

The θ and ϕ equations are (after a little tidying)

$$R\dot{\phi}^2\sin\theta\cos\theta - g\sin\theta = R\ddot{\theta} \qquad \text{and} \qquad mR^2\dot{\phi}\sin^2\theta = \text{const.}$$

(b) The ϕ equation tells us that the z component of angular momentum, $\ell_z = mR^2\dot{\phi}\sin^2\theta$, is constant.

(c) If ϕ is constant, the θ equation reduces to $-g\sin\theta = R\ddot{\theta}$, which is the equation for a simple pendulum. That is, in this case, the pendulum swings in a single vertical plane, $\phi = \phi_o$, just like a simple pendulum.

(d) If we replace $\dot{\phi}$ by $\ell_z/(mR^2\sin^2\theta)$, the θ equation becomes

$$R\ddot{\theta} = k\frac{\cos\theta}{\sin^3\theta} - g\sin\theta, \tag{xiii}$$

where k denotes the positive constant $k = \ell_z^{\,2}/m^2R^3$. Now θ can remain constant if and only if $\ddot{\theta} = 0$, which requires that θ satisfy $k\cos\theta = g\sin^4\theta$. This equation can only be satisfied if $0 < \theta < \pi/2$. (If $\pi/2 < \theta \le \pi$, the left side is negative while the right is positive.) If we vary θ from 0 to $\pi/2$, the left side decreases steadily from k to 0 while the right side increases steadily from 0 to g. Therefore there is exactly one value $\theta = \theta_o$ at which the angle θ can remain constant. In this motion the string of the pendulum traces out a vertical cone of half angle θ_o.

(e) We can rewrite the θ equation (xiii) as $R\ddot{\theta} = f(\theta)$, where $f(\theta) = k(\cos\theta/\sin^3\theta) - g\sin\theta$. Now, at the "equilibrium" value $\theta = \theta_o$ we know that $f(\theta_o)$ is zero. Thus if θ is close to the equilibrium value, $\theta = \theta_o + \epsilon$, we can write $R\ddot{\epsilon} \approx f'(\theta_o)\epsilon$, and by differentiating you can check that $f'(\theta)$ is the sum of three terms, all negative in the range $0 < \theta < \pi/2$. Therefore the equation of motion has the form $\ddot{\epsilon} = (\text{negative constant})\epsilon$, and θ executes simple harmonic motion about θ_o. The motion of the bob is uniform motion in a horizontal circle with a superposed small sinusoidal motion in the $\hat{\boldsymbol{\theta}}$ direction.